Matura Examinations 2023 – Mathematics

Classes: 4Be, 4LW
Teachers: BtT, PrG

Note:	You have four hours to complete the examination.
	Begin each question on a new sheet of paper.
Permitted materials:	TI-*n*spire CX calculator (in 'press-to-test' mode)
	The *Fundamentum Mathematics and Physics*, without notes
	English-German dictionary

All questions labelled with the symbol ✍ are to be solved **by hand**. For these questions, only the basic functions of your calculator are permitted. To attain full marks in these questions, you should not use commands such as dotP, nSolve, polyRoots or the numerical calculation of derivatives or integrals.

In general, the graphics window of your calculator should only be used to visualise the graphs of functions.

Question 1: Vector Geometry (13 marks)

The points $A(6|0|4)$, $B(0|6|4)$, $C(-6|0|4)$ and D lie in plane Π_1, and form the square base of a right pyramid $ABCDS$ whose summit point is $S(0|0|1)$. The points A, B and S all lie in the plane Π_2 (one of the sloping faces).

(a) Mark these points onto the 3D coordinate system provided on page 8. Find the coordinates of the point D and describe the particular position of the plane Π_1. Finally, create a Cartesian Equation for the plane Π_1. (2 P.)

(b) Demonstrate that the triangle ABS is isosceles. (1 P.)

(c) Find a Cartesian equation for the sloping plane Π_2.
(As a check for your final result: $\Pi_2 : x + y - 2z + 2 = 0$) (2 P.)

(d) Calculate the size of the angle θ, between two adjacent ridges[1] that meet at summit point S. (1.5 P.)

(e) The line that passes through the points $P(13|7|-7)$ and $Q(8|4|-2)$ cuts plane Π_2 at the point T. Calculate the coordinates of point T. Can we be sure that this point lies on the triangular face of the pyramid? Justify your decision. (3 P.)

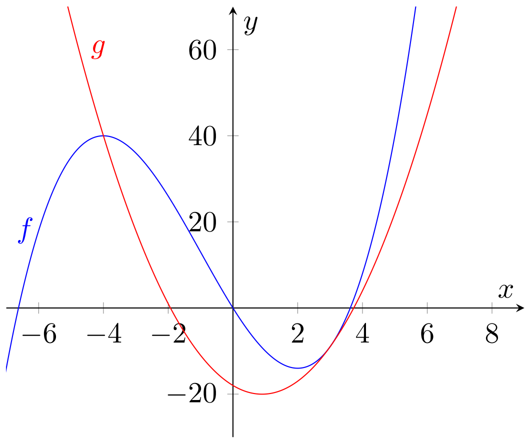
(f) The summit point S is now able to move freely, along the z-axis. We wish to find the positions, S^*, so that the sloping, triangular face ABS^* becomes the angular bisector of the planes Π_1 and Π_2. Find these positions for S^*, and state which one of them lies inside the original pyramid $ABCDS$. (3.5 P.)

[1] ridges = Kanten

Question 2: Calculus (12 marks)

The functions f and g are given as:

$$f(x) = \frac{1}{2}x^3 + \frac{3}{2}x^2 - 12x \quad \text{and} \quad g(x) = \frac{5}{2}x^2 - \frac{9}{2}x - 18$$



✍(a) For the graph of function f, find the zero points, maximum point, minimum point and the point of inflexion. You should confirm, by calculation, the type of each the two extreme points. (7 P.)

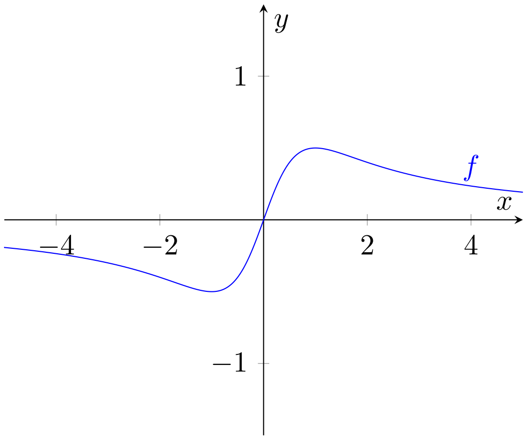
✍(b) Find an equation for the tangent to the graph of f where $x = -2$. (1.5 P.)

(c) Show that the graphs of functions f and g are touching where $x = 3$. (1 P.)

✍(d) The graphs of f and g intersect where $x = -4$, and thus create a bounded region between them. Calculate the area of this bounded region. (2.5 P.)

Question 3.1: Calculus (8 marks)

The function f may be expressed as: $f(x) = \dfrac{x}{x^2+1}$.



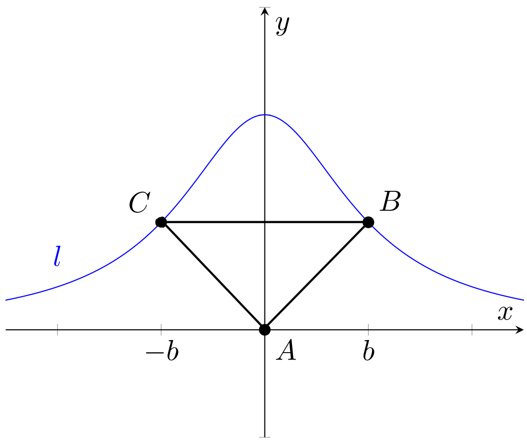
(a) Does the graph of f possess any form of symmetry? If yes, state what type, justifying your decision with a calculation. (1 P.)

(b) Find an equation for the horizontal asymptote of the graph of f, and explain why there is no vertical asymptote. (1.5 P.)

(c) Show that $f'(x) = \dfrac{-x^2+1}{x^4+2x^2+1}$ represents the derivative function of f. (1.5 P.)

(d) The graph of function f and the curve with equation $g(x) = \dfrac{1}{2}x^2$ intersect at $S(1|0.5)$. Calculate the angle α between the graphs at this intersection point. (2 P.)

(e) A new function l is defined as follows $l(x) = \dfrac{1}{x^2+1}$.

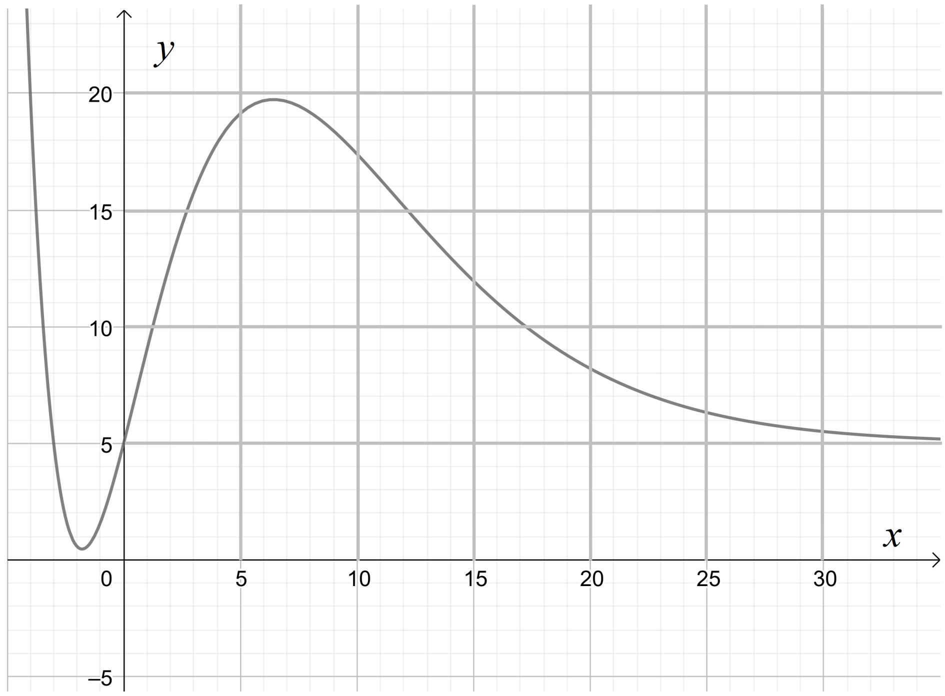


Drawn into the above diagram is an isosceles triangle with vertices $A(0|0)$, $B(b|l(b))$ and $C(-b|l(-b))$ where $b > 0$.

The rotation of triangle ABC around the y-axis creates a simple cone. What range of values are possible for the cone's volume? (2 P.)

Question 3.2: Calculus (4 marks)

A section of the graph of function k is represented in the diagram below. Note: All extreme points and points of inflexion can be found in this section. Also, the graph of function k possesses the horizontal asymptote $y = 5$.



The next two questions refer to the graph of the derivative function k'. You should justify your answers to both of them.

(a) Give approximate values of x where the graph of the derivative function, k' has any maximum or minimum points. (1 P.)

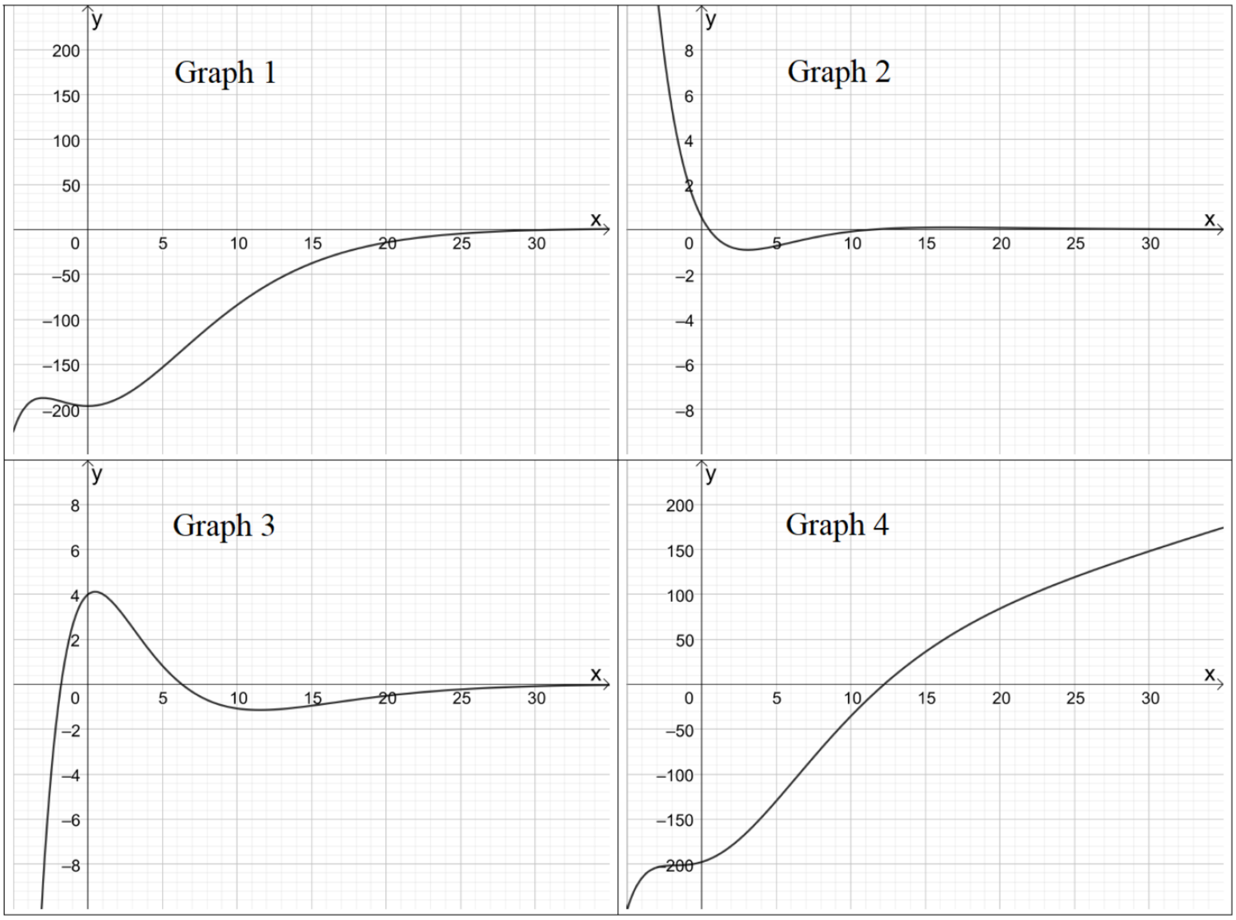
(b) Find the value of $\lim_{x \to \infty} k'(x)$. (1 P.)

The next two questions refer to the integral $\int_a^b k(x)\,dx$.

(c) Write down an approximate value for $\int_5^{10} k(x)\,dx$. (1 P.)

Continued on the following page.

(d) Which of the following four graphs could represent an integral function K for k? Justify your decision. (1 P.)



Question 4: Combinations and Probability (13.5 marks)

The three parts of this question are independent of each other.

4.1 Whale-watching

You reserve a boat trip in order to observe whales at liberty in the ocean. However, the probability that you see any whales during a trip is only 60%. The tour operator, therefore, offers a second (free) boat trip to those who were unable to see any whales during the first trip.

(a) Given this offer, what is the probability that you can expect to see one, or more, whales? (1.5 P.)

(b) What is the minimum number of boat trips that you would need to make to be at least 99% sure of seeing at least one whale? (2 P.)

4.2 Volleyball

Team A and Team B play a match of Volleyball. The winning team is the first one to win two sets. Team A is considered to be the stronger team: the probability that they win a set against Team B is considered to be 55%.
However, if Team A win a set against Team B their probability of winning the next set rises by five percentage points. On the other hand, if Team A loses a set, their probability of winning the next set drops by five percentage points.

(a) What is the probability that Team A will win this "best of three" match? (3 P.)

(b) A supporter of Team B is ill at home. He receives a message telling him that Team B has just won a set, and the game is continuing. Given the above information, what is the probability that Team B will win the Volleyball match? (3 P.)

4.3 A Game of Draughts

Draughts (pronounced "Drarfts") is a game of strategy between two players. Each player receives 12 identical 'counters' (black or white). To begin a game, these 'counters' must be placed in the arrangement shown in the diagram.

(a) If, however, all 32 black squares were available, in how many different positions could a player place her 12 white 'counters'? (1 P.)

(b) If all 32 black squares were available, in how many different positions could the 12 black 'counters' and the 12 white counters be placed? (One example is shown above) (1.5 P.)

(c) Twelve players take part in a Draughts' Tournament. At the end, every player may choose a prize[2] Everyone's choice will be successful! Once the Tournament is complete, the final Results' Table is published, and the players must then make their choice(s) from five different cinema films. How many different arrangements of cinema film selections are possible from this Results' Table? (1.5 P.)

[2]First, second and third-placed players may choose two tickets to two different cinema films. The remaining nine players may chose (only) one ticket to a single cinema film.

Question 5 (12.5 marks)

5.1 Trigonometry

Two farmers own land on a (quadrilateral) field. Each farmer's land is separated by a fence built from point A to point B, and from point B to point C.
This separation (Diagram 1) needs to be modified: a line of separation will be created from point C directly to a new point D (see Diagram). However, this new line must not change the area of the field that each farmer owns.
In order to construct this new (straight) fence, the distance $\overline{AD}$ must be calculated. The following information is already available:
Length: $\overline{AB} = 282\,\mathrm{m}$; Length $\overline{BC} = 145\,\mathrm{m}$.
Angle $\gamma = 52°$ and angle $\varepsilon = 74°$.

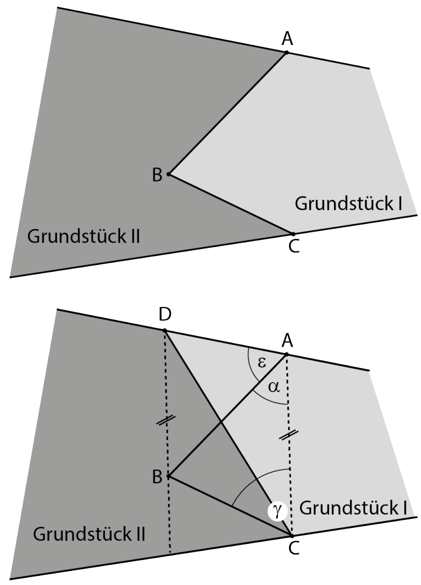


Figure 1: Diagram

(a) Calculate the size of angle α. (1.5 P.)

(b) "In Diagram, the dotted lines AC and BD are parallel." Justify this statement. (1 P.)

(c) Calculate the distance from point A to point D. (2.5 P.)

5.2 Logarithmic Equation

✍ Solve the following equation:

$$\log_2(x-6) + \log_2(2x) = 5$$

(3 P.)

5.3 Exponential Decay

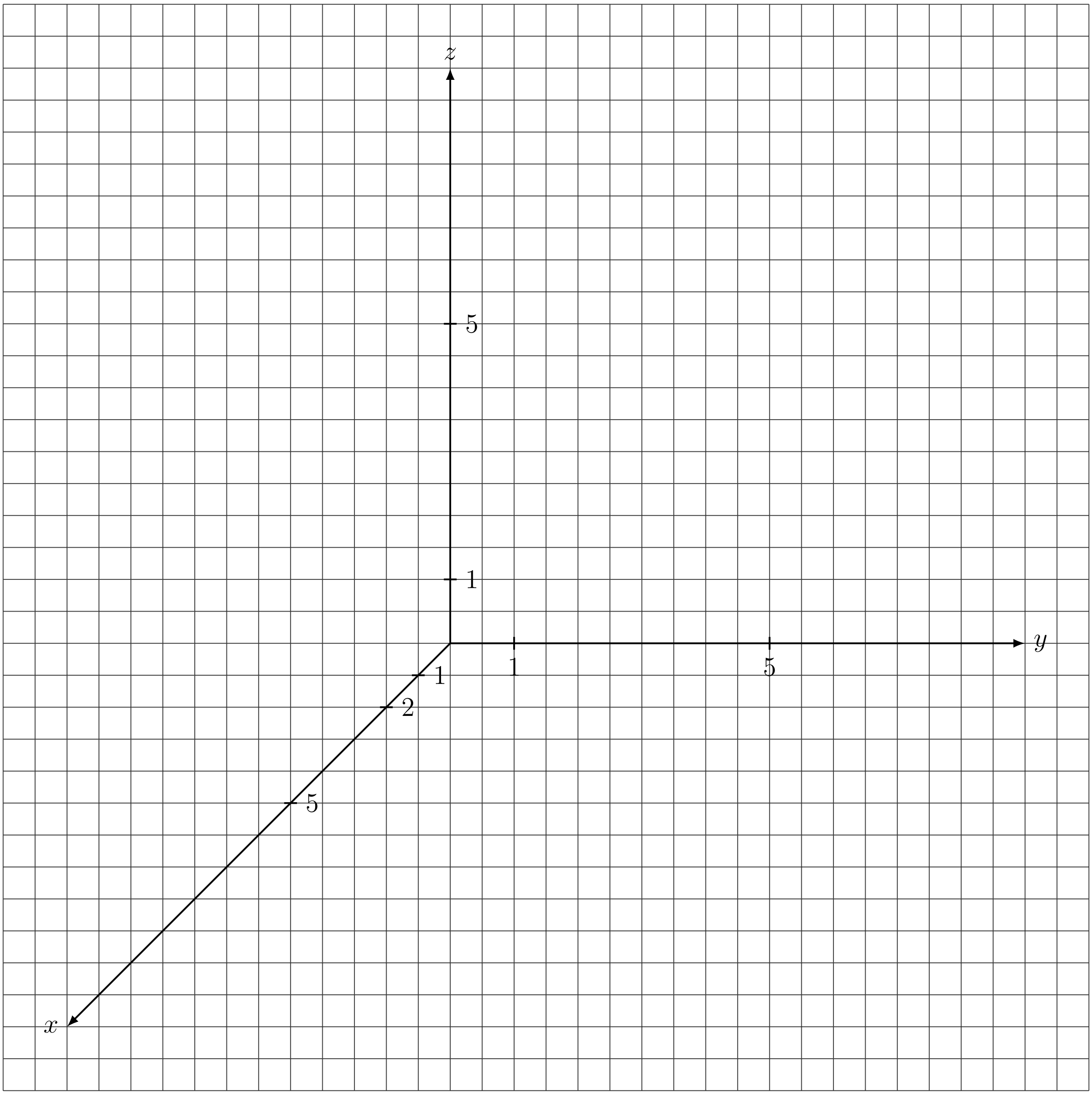
A tablet contains an active substance which decays in the human body according to an exponential process. Immediately after swallowing one tablet, $800\,\mathrm{mg}$ of the active substance are released into the body. Ten hours later, exactly $60\,\mathrm{mg}$ of this active substance remain.

(a) What quantity of the active substance is still present in the body 16 hours after a tablet has been swallowed? (1.5 P.)

(b) How long will it take until the quantity of active substance present in the body has fallen to exactly half of the starting value? (1 P.)

A patient is due to be given a new medicine which contains $400\,\mathrm{mg}$ of an active substance that decays exponentially in the human body according to the function $m(t) = m_0 \cdot 0.65^t$, where t is the time, in hours, which has passed since the medicine was swallowed.
The 'tablet instructions' given to the patient are, as follows: Take one tablet at 8h 00: Take two tablets at 14h 00: Take one tablet at 20h 00.

(c) What quantity of this active substance will be present in the patient's body at 8h 00 the following morning before swallowing the next tablet? (2 P.)


z
5
1
y
1
5
1
2
5
x