

## Matura Examinations 2023 – Mathematics

Class: 4AWe (Profile A)

Teacher: ScR

- Note: You have four hours to complete the examination.  
Begin each question on a new sheet of paper.
- Permitted materials: TI-*n*spire CAS calculator (in 'press-to-test' mode)  
*Formulæ, Tables and Concepts* from Orell Füssli Verlag, without notes  
English-German dictionary

The maximum number of marks which can be attained in this exam is 53. **Questions 1–3** (printed on coloured paper) are to be solved **without a calculator**. These questions comprise 26.5 marks, or exactly 50% of the whole exam.

Once you have handed in your answers to these questions (as well as the coloured question sheets), you may take your calculator from the supervisor.

### Question 1: Analysis (6.5 marks)

(a) Consider the polynomial function

$$p(x) = 3x^5 - 5x^4 + 7x$$

Find the coordinates of all points of inflection of the graph of  $p$ . (2.5 P.)

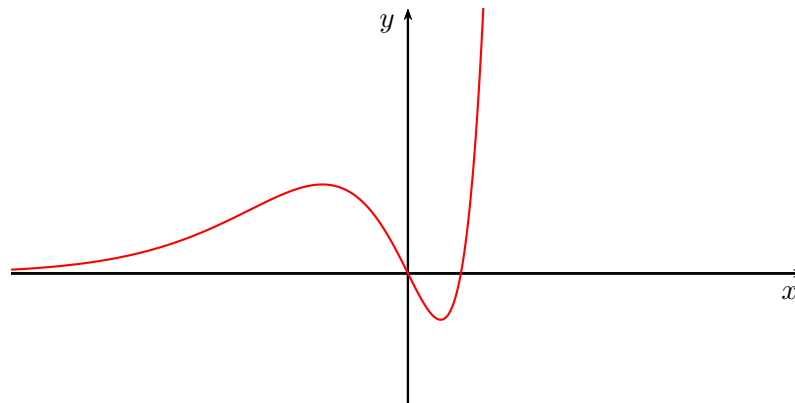
(b) Now consider the rational function

$$f(x) = \frac{x^6 - 5x^3}{2x^4 + 2x^2 - 12}$$

- i. Find the domain of the function  $f$  and discuss the nature of the function's singularities. You do not need to determine whether there is a 'change of sign' at any poles. (2.5 P.)
- ii. For large values of  $x$ , the graph of  $f$  gets closer and closer to the graph of a polynomial function. Which polynomial function is it? (1.5 P.)

## Question 2: Analysis (13.5 marks)

Shown below is (part of) the graph of the function  $g(x) = (x^2 - 2x)e^{0.5x}$ .



- (a) Explain how we know that the graph of  $g$  has only two  $x$ -intercepts. (0.5 P.)
- (b) Show that the derivative of  $g$  can be written as follows:

$$g'(x) = \left( \frac{x^2}{2} + x - 2 \right) e^{0.5x} \quad (1.5 \text{ P.})$$

- (c) Denote by  $P$  the  $x$ -intercept of the graph  $y = g(x)$  which lies to the right of the origin, and by  $\ell$  the tangent to the graph at the point  $P$ . Where does  $\ell$  intersect the  $y$ -axis? Find the angle formed between  $\ell$  and the  $y$ -axis at this point.

**Note:** Since you don't have a calculator, you won't be able to find a numerical value for the required angle. Instead, you should give a mathematical expression for the angle. (2 P.)

- (d) Find a polynomial function whose graph also intersects the  $x$ -axis at the origin and at the point  $P$ , and has the same gradient as the graph of  $g$  at these two points. (4.5 P.)
- (e) Calculate the area of the region in the second quadrant which is enclosed between the graph of  $g$  and the  $x$ -axis.

**Note:** If you cannot find an anti-derivative of the function  $g$ , use the (incorrect) anti-derivative  $G(x) = (3x^3 - 2x^2 + 6)e^{0.5x}$  instead. (5 P.)

## Question 3: Complex Numbers (6.5 marks)

The function

$$f(z) = z - \frac{z^3 - 1}{3z^2} \quad \text{for all } z \in \mathbb{C} \setminus \{0\}$$

is very famous in the area of 'complex dynamics' and can be used to create an intricate fractal. We will consider more mundane aspects of this function:

- (a) Calculate  $f(1 + i)$ . Give your answer in the rectangular form  $a + bi$ . (2 P.)
- (b) Find all complex numbers  $z$  which satisfy the equation  $f(z) = 0$ . Give your answers in the polar form  $z = re^{\theta i}$ . (2 P.)
- (c) Show that for every non-zero complex number  $z$ ,

$$f\left(e^{\frac{2\pi i}{3}} \cdot z\right) = e^{\frac{2\pi i}{3}} \cdot f(z) \quad (2.5 \text{ P.})$$

For the remaining questions 4–6 you may use your calculators.

### Question 4: Vector Geometry (10 marks)

Consider the sphere  $\mathcal{S}$  with centre  $M(-1, 0, 6)$  and radius 7, as well as the points  $A(3, -6, -3)$  and  $B(5, -2, 3)$ . The point  $B$  lies on  $\mathcal{S}$ .

- (a) Find the  $x$ -intercepts of the sphere  $\mathcal{S}$ . (1 P.)
- (b) Does the point  $A$  lie on, inside or outside the sphere  $\mathcal{S}$ ? (1 P.)
- (c) Denote by  $\ell$  the straight line passing through the points  $A$  and  $B$ , and by  $\Gamma$  the plane which lies tangent to  $\mathcal{S}$  at the point  $B$ . Find the angle between  $\ell$  and  $\Gamma$ . (2 P.)
- (d) Find the distance between  $\mathcal{S}$  and the plane with equation  $12x - 5y = 157$ . (1.5 P.)

The point  $P(-5, -6, 18)$  lies outside the sphere  $\mathcal{S}$ . Consider the set of straight lines which pass through the point  $P$  and are tangent to the sphere  $\mathcal{S}$ . These tangent lines form an (infinite) cone, which we will denote by  $\mathcal{C}$ . Also, the set of all points where the tangent lines touch the sphere  $\mathcal{S}$  forms a circle, which we will denote by  $\mathcal{K}$ .

**Note:** The following questions can be solved using various methods from elementary geometry or vector geometry. You may use any method you choose.

- (e) Show that the opening angle of the cone  $\mathcal{C}$  is equal to  $60^\circ$ . (1.5 P.)
- (f) Find the centre and the radius of the circle  $\mathcal{K}$ . (3 P.)

### Question 5: Sequences and Series (5 marks)

Reporting on environmental affairs, a journalist makes the following announcement:

*“Scientists have calculated that if the world’s annual gas consumption stays constant at its current level, worldwide gas reserves will only last for another 65 years. In response, the UN General Assembly has suggested reducing the world’s gas consumption successively as follows: each year, the consumption should be 2% lower than the previous year.”*

Suppose that the worldwide gas consumption is currently  $a$  cubic metres per year, and that the scientists’ calculations are correct.

- (a) How many cubic metres of gas would be consumed in total during the next 30 years if in the first year the consumption is still  $a$  cubic metres, but in the following years the gas consumption is reduced according to the UN’s suggestion? Is it true that after 30 years, approximately two thirds of the world’s gas reserves would still remain? (2.5 P.)
- (b) Show using appropriate calculations that if the world reduces its gas consumption by 2% every year as the UN suggests, then its gas reserves will never run out. (1 P.)
- (c) Under pressure from various countries, the UN is forced to weaken its suggestion. It still wants to reduce the world’s gas consumption by a certain fixed percentage every year, but it decides that 2% is too high. Find the smallest percentage for the yearly reduction which would still ensure that worldwide gas reserves never run out. (1.5 P.)

## Question 6: Probability (11.5 marks)

To demonstrate the importance of mathematics in modern computer science, a teacher decides to spend a semester working on a Machine Learning project with her class. She divides the class into two groups. Each group is given the task of writing an algorithm which can distinguish cats from dogs. Given a photo as its input, the algorithm should return either "CAT" or "DOG" as its answer. No other output is allowed. At the end of the semester, the groups present their programs to the rest of the class. As part of the presentation, each group demonstrates its algorithm using a test set of input photos provided by the teacher.

### GROUP 1

Group 1 produces an algorithm which correctly identifies 90% of all cats it is shown, and 80% of all dogs. The teacher gives this group a test set containing 10 cat photos and 10 dog photos.

- (a) In how many different ways can the 20 photos be presented to the algorithm
- i. in total? (0.5 P.)
  - ii. if all the cat photos should appear one after the other? (1 P.)
- (b) What is the probability that the algorithm identifies all 20 photos correctly? (1 P.)
- (c) What is the probability that the algorithm identifies exactly 18 photos correctly?  
**Hint:** Consider different possible cases. (2.5 P.)
- (d) The algorithm identifies the first photo as a cat. What is the probability that the algorithm is right? (2 P.)

At the end of their presentation, a pupil comments that it would be interesting to see what the algorithm would do if presented with photos of other animals, or even people. The teacher likes the idea, and creates a new test set containing 17 photos of animals and 3 photos of teachers. Group 1 presents the 20 photos to its algorithm in random order.

- (e) Calculate the probability that the third photo is of a teacher. (1 P.)
- (f) Calculate the probability that among the first 8 photos presented to the algorithm, exactly 2 are of teachers. (1 P.)

### GROUP 2

Group 2 produces an algorithm which correctly identifies all cats as cats, but unfortunately also identifies 40% of all dogs as cats.

- (g) Suppose the teacher also gives this group a test set containing 10 cat photos and 10 dog photos. What is the probability that the algorithm achieves a success rate of 90% or higher? (1 P.)
- (h) Now suppose the teacher gives this group a test set containing  $N$  photos. How many cat photos would need to be in the test set so that Group 2 could expect its algorithm to achieve a success rate of at least 90%? (1.5 P.)