Matura Examinations 2024 – Mathematics

Classes: 4A(W), 4Ae (Profile A)
Teachers: KrD, PrG

The maximum number of marks is 60. **Questions 1–6** (printed on coloured paper) are to be solved **without a calculator**. These questions comprise 48.5 marks, which is approximately 80% of the whole exam.

Once you have handed in your answers to questions 1–6 (as well as the coloured question sheets), you may take your calculator from the supervisor and use it to solve question number 7 (Stochastics).

Note:	You have four hours to complete the examination.
	Begin each question on a new sheet of paper.
Permitted materials:	*Formulæ, Tables and Concepts* from Orell Füssli Verlag, without notes
	English-German dictionary
	Questions 1–6: no calculator
	Question 7: Calculator permitted (4A(W): TI-nspire CAS in Press-to-Test-Mode, 4Ae: TI-30X Pro MathPrint)

Dennis Krüger and Gérald Prétôt wish you success.

Question 1: Vector Geometry (11 marks)

Note: The following part questions may be solved in different ways. Choose the one that is most effective/shortest. You may use vector geometry as well as trigonometry to solve parts.

The vectors $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ span a parallelepiped (*Spat*) from the origin O.

(a) Use calculatations to argue that the parallelepiped spanned by the vectors $\vec{a}$, $\vec{b}$ und $\vec{c}$ is actually cuboid, i.e. the edges are perpendicular to each other and then calculate the volume of this cuboid. (2 P.)

(b) Test by calculation whether the following claim is true:

The measure of the angle between the line h passing through the vertex Q furthest away from the origin O and through the z-axis at 4, and the diagonal OQ is $\alpha = 60°$. (2.5 P.)

(c) Calculate the centre and radius of the sphere K_1 given as $K_1 : x^2 + y^2 + z^2 - 4x + 3y - 3z = 0$ and check whether K_1 passes through all vertices of the cuboid. (2 P.)

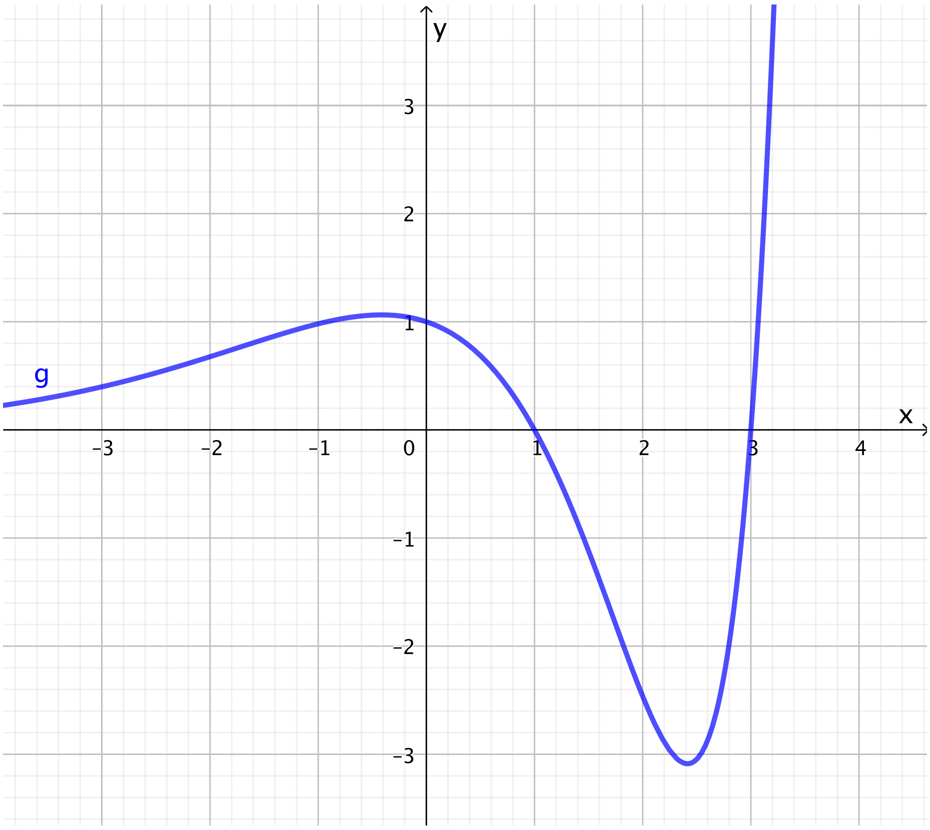
(d) Now consider the sphere K_2 with radius $r = 2.5$ and centre M, the midpoint of the cuboid.

i. What is the distance of the plane $E : 3x + 2y - 6z = 43$ from the sphere K_2? (2 P.)

ii. The sphere K_2 intersects with all rectangular faces of the cuboid in a circle. Calculate the radius of the largest such circle. (2.5 P.)

Question 2: Calculus (8 marks)

The figure below shows part of a graph of function g with the definition $g(x) = (ax^2 + bx + c) \cdot e^x$.
It intersects with the coordinate axes at integer coordinates. a, b and c are real number parameters and e is the Euler number.



(a) Given the above, calculate the values of the parameters a, b and c, and thus the concrete definition of the function $g(x)$. (3 P.)

(b) The figure only shows part of the graph of g. Discuss whether the graph intersects with the x-axis outside of the shown range. (0.5 P.)

Note: If you cannot find the equation for g, continue using the substitute $g(x) = \left(\frac{1}{3}x^2 - \frac{4}{3}x + 1\right) \cdot e^x$.

(c) Calculate the size of the area bounded by the graph of g and the x-axis, which lies below the x-axis. Perform your calculations from scratch without relying on the formulary. (3 P.)

Note: If you could not find the antiderivative of g, continue using $G(x) = e^x \cdot \left(\frac{x^2}{3} + 3 - 2x\right)$.

(d) Check if the following claim is true:

The unbounded area between the graph of g and the x-axis over the interval $I = (-\infty; 1]$ is larger than the area calculated in the previous part question. (1.5 P.)

Question 3: Calculus (7 marks)

Copnsider the following real valued rational function f:

$$f(x) = \frac{x^6 - 16x^2}{2x^2 - 4x}$$

(a) Analyse, whether the graph of f is symmetrical about the origin or about the y-axis. (1 P.)

(b) Where are there discontinuities of the graph of f? (1 P.)

(c) Calculate all zeros of f. (1.5 P.)

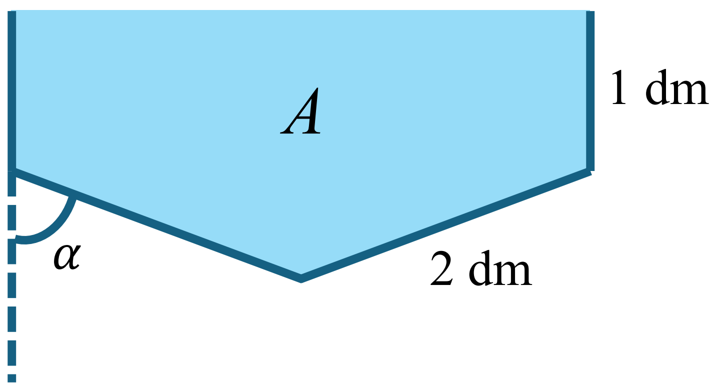
(d) Determine the equation for the assymptotic graph $A(x)$ to the graph of $f(x)$. (2 P.)

(e) Explain, why the graph of f cannot have any vertical assymptotes. (0.5 P.)

(f) **Without calculation**, using the result of the previous two part questions, discuss whether the graph of f can have three inflexion points. (1 P.)

Question 4: Optimisation, Trigonometry (6 marks)

Peter uses a rectangular sheet of metal, six decimetres by twelve meters to form a gutter (*Wasserabflussrinne*) - see the picture of the cross section.
Due to regulations the two sides must be perpendicular to the ground and have a height of exactly one decimetre. Only the angle α may vary.



(a) Express the area of the cross section A as a function of the angle α, i.e. find a formula for $A(\alpha)$.
Note: Extend the given sketch as needed. (2 P.)

(b) There is an angle α between zero and ninety degrees for which the gutter can hold the largest amount of water. Calculate the exact measure of this angle α. (4 P.)
Note: If you were unable to solve part (a), use the (incorrect) formula
$A(\alpha) = 3 \cdot \sin(\alpha) \cdot \cos(\alpha) + 3 \cdot \sin(\alpha)$.

Question 5: Sequences and Series (6.5 marks)

Given the sequence (a_n) with general member

$$a_n = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} \qquad \text{(where } n \in \mathbb{N}, n \neq 0)$$

(a) Show that the sequence (a_n) has no negative members and argue that the sequence is monotonically decreasing. (3 P.)

(b) Calculate the limit a of the sequence. (0.5 P.)

Let us consider the series $s_n = \sum_{k=1}^{n} a_k$ belonging to the sequence (a_n).

(c) Calculate $s_3 = \sum_{k=1}^{3} a_k$. (1 P.)

(d) Use mathematical induction to prove the following formula:

$$s_n = \sum_{k=1}^{n} a_k = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}, \quad \text{for all } n \in \mathbb{N} \qquad \text{(2 P.)}$$

Note: Make sure you document all formal steps of the mathematical induction.

Question 6: Conic Sections/Complex Numbers (10 marks)

We consider a subset ε of the set of the complex numbers $\mathbb{C}$, for which the following holds:

$$\varepsilon = \left\{ z \in \mathbb{C} \;\middle|\; \frac{-\left(\mathrm{Re}(z)\right)^2}{\mathrm{Im}(z) - \frac{3}{4}} = 4 \cdot \mathrm{Im}(z) + 3 \right\},$$

where $\mathrm{Re}(z)$ is the real part and $\mathrm{Im}(z)$ the imaginary part of the complex number z.

(a) Show, that ε forms an ellipse in the complex (Argand) plane and provide its cartesian equation. (2 P.)

Note: If you could not find the cartesian equation for ε, continue using $\varepsilon\colon 1 = \frac{x^2}{\frac{9}{4}} + \frac{y^2}{\frac{9}{16}}$.

(b) Find the complex numbers $s_1, s_2, s_3, s_4 \in \mathbb{C}$, that correspond with the vertices of the ellipse ε. (0.5 P.)

(c) Determine the complex numbers $f_1, f_2 \in \mathbb{C}$, that correspond with the foci of the ellipse ε. (1 P.)

(d) Calculate all intersections of the ellipse ε with the unit circle of the complex (Argand) plane. Give the corresponding numbers in cartesian (sum) form and as polar coordinates. (3 P.)

Independent of the the above part questions:

(e) Calculate all fixed points of the complex function $f(z) = z^2 + z - (5 + 12i)$ in cartesian (sum) form. (3.5 P.)

Note: Only after handing in your answers to questions 1–6 (as well as the coloured question sheets), you may take your calculator from the supervisor and use it to solve question number 7 (Stochastics).

Question 7: Stochastics (11.5 marks)

We use fair tetrahedrons with faces with numbers 1–4 as dice. The number that the tetrahedron lands on is the one that we count.

In a *try*, we use three identical tetrahedra and note down the numbers.

A *pair* is the event of rolling exactly two identical numbers plus one different from these two in a single *try*.

(a) What is the probability in **a single try** of achieving:

a1. three times the number 4? (0.5 P.)

a2. exactly one 1, one 2 and one 3? (1 P.)

a3. three numbers different from each other? (0.5 P.)

a4. *a pair* - show that the probability is $\frac{9}{16}$. (1 P.)

(b) We now perform a sequence of **ten** *tries* of the three tetrahedra.

What is the probability within such a sequence of ten *tries* of rolling

b1. no *pairs* at all, (0.5 P.)

b2. exactly three *pairs*, (1 P.)

b3. *a pair* at least twice? (1 P.)

b4. What is the *expected value* of the number of rolled *pairs* within such a sequence of ten *tries*? (1 P.)

(c) If we perform an **arbitrary number** of *tries* of the three tetrahedra,

c1. what is the probability of rolling a first *pair* in the twelveth *try* (1 P.)

c2. what is the least number of *tries* necessary to roll at least *one pair* with a certainty of at least 99.9%? (1.5 P.)

(d) We now use **two** tetrahedra that cannot be distinguished from each other from the outside. One of the two is **manipulated** such that the probability of rolling a 4 is twice that of any of the other numbers.

We now roll these two tetrahedra *once*.

d1. What is the probability of getting **exactly one** 4? (1.5 P.)

d2. If exactly one 4 was rolled, what is the probability for it stemming from the **manipulated tetrahedron**? (1 P.)